LIMIT THEOREMS FOR FUNCTIONALS OF MARKOV PROCESSES AND RENORMALIZABLE STABLE FIELDS

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ABSTRACT

We study the limiting distribution of the amount of charge left in some set by an infinite system of charged Markovian particles, when the charge distribution belongs to the domain of attraction of a symmetric α-stable law. The limits are symmetric α-stable generalized random fields. Their multiple integrals are built in a similar manner. We also study the renormalizability of these families of random fields and use the construction to simulate stable fields on R^1 and R^2.

1. INTRODUCTION

Stable processes have been studied intensively in recent years. Samorodnitsky and Taqqu (1990), Weron (1984), Rosinski and Woyczynski (198j), Rosinski (1986), Kallenberg and Szulga (1989), Szulga (1992), Maejima (1990), Janicki and Weron (1991), are some of the references related to this work. Because all these authors work on an abstract measurable space, or with specific models on R^1, we were motivated to define families of stable random fields on $R^d, d \geq 1$, and of generalized stable fields on the Sobolev space $\mathcal{S}_d$. We build the fields as limits of sums of functionals on paths of Markov processes and show that using this construction one can easily analyze some of the properties of the fields in the limit.

Motivated by work of Maejima (1990) on self-similar stable processes, we choose to study the renormalizability of the families of stable fields and their functionals. The property of renormalizability of families of random fields was defined in Adler and Epstein (=Feldman) (1987) and Epstein (1989), as an extension of the property of self-similarity to families of fields. These authors discuss the renormalizability of families of Gaussian fields. Here we show that stable fields built from self-similar Markov processes are renormalizable.

Our method of construction of the stable fields also allows us to simulate these fields on a computer and display the surfaces to see what they look like. Our moti-

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vation in this direction comes from work of Janicki and Weron (1991) on simulation of stable processes (all on R^1).

We now look at our results from a somewhat different point of view. In recent years, much attention has been given to the description of infinite systems of particles moving according to some law (usually Markovian). Some of these papers e.g., Adler and Epstein (1987) and Adler (1989), deal with particle systems which behave as follows: Initially (at time zero) a number of independent particles pop into existence at locations within the space R^d, according to a Poisson point process with intensity λ. (Actually the initial distribution of the particles was created differently in the above papers, but one could use Poisson point processes instead.) The particles then move according to some Markov law. A positive or negative charge is assigned to each particle initially, according to a Rademacher random variable. The charge of each decays exponentially with time. The number of particles in the system is then set to infinity and the limiting distribution of the charge left by the system in a set in R^d, after all the particles have lost their charge, is studied. The limiting field, which is indexed by sets, or more generally by functions, was shown to have a Gaussian distribution in Adler and Epstein (1987). In Feldman and Rachev (1993), the authors obtain limiting fields that have sub-Gaussian, Laplace and other distributions, by changing the initial distribution of the particles appropriately.

The second aim of this work is to answer the question, "What happens to the limiting charge distribution if the initial charge of each particle follows a symmetric α-stable law?"

As in the above-cited work, our main tools are limit theorems for U-statistics. Here we use results found in Szulga (1992) on the convergence of resampled U-statistics to multiple stable integrals.

This paper is organized as follows: In Section 2, we present our construction of generalized stable random fields and their multiple integrals. The proofs are given in Section 3. In Section 4, we show how to construct stable fields on R^d. In Section 5 we study the renormalizability of the families of stable fields and their multiple integrals. Section 6 is devoted to simulation results for some stable processes and fields on R^1 and R^2.

2. STABLE RANDOM FIELDS AND THEIR FUNCTIONALS.

Let us now define the particle system of the Introduction precisely. On an arbitrary probability space $(\Omega, \mathcal{F}, \mathrm{P})$, take n independent symmetric Markov processes $V_1, V_2, \ldots, V_n$ with values in R^d, each process starting according to some finite initial measure m. Let $p_t(x,y)$ be their common transition density function; $p_t(x,y) = p_t(y,x)$, $x, y \in R^d$, $t \geq 0$ and $\int_{R^d} p_t(x,y)dy = 1$, for each $x \in R^d$. The corresponding Green's function is given by

$$g(x,y) \equiv g^1(x,y), \quad g^\theta(x,y) = \int_0^\infty e^{-\theta t} p_t(x,y)dt, \tag{1}$$

If V is one of the processes in the sequence, then it can be thought of as describing the motion of a particle in the system. We now discuss the charge assigned to each of the n particles. Let the probability space be rich enough to support the sequences of i.i.d. random variables, $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,n}$, which are independent of the V's and for which

$$P(Y_{n,i} > t) = \frac{1}{c_\alpha n} \int_t^\infty \frac{dx}{x^{1+\alpha}} = \frac{1}{\alpha c_\alpha n t^\alpha} \text{ for } t > (\alpha c_\alpha n)^{-1/\alpha}, \tag{2}$$

where $c_\alpha = \int_{R \setminus \{0\}} (1 - \cos x)dx/x^{1+\alpha}$. Let $\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots$ be a sequence of i.i.d. Rademacher variables (on the same probability space), i.e., $P(\sigma_i = 1) = P(\sigma_i = -1) = 1/2$, which are independent of the V's as well as the Y's. Assign to each

particle V_i a charge $\sigma_i Y_{n,i}$. We will call $\sigma_i Y_{n,i}$ the charge associated with the Markov particle V_i. The particle charges clearly belong to the domain of attraction of a stable law.

We now describe the evolution of the system in time. When particle i with charge $\sigma_i Y_{n,i}$ at time t passes through a point x in the space R^d, it leaves there a charge $\sigma_i Y_{n,i} e^{-t}$. Let $A \in \mathcal{B}(R^d)$ be a Borel set in the space R^d. We would like to find the amount of charge left in the set A after all particles have lost their charge and in the limit of increasing initial particle density, i.e., we are interested in finding the limiting distribution, as $n \to \infty$, of the normalized sum

$$\Phi_n(A) := \sum_{i=1}^{n} \int_0^\infty \sigma_i Y_{n,i} e^{-t} 1_A(V_i(t))dt.$$

More generally, define the bilinear form

$$\langle f, h\rangle \equiv \langle f, f\rangle_m := \int_{R^{3d}} m(da)dx\,dy\, f(x)g^2(a,x)g(x,y)f(y) \tag{3}$$

where g and g^2 were defined in (1). Let $S_d \equiv S_d(g)$ be the Sobolev space of C^∞ functions of finite norm, where the inner product in S_d is given by (3). For f in S_d we define the functional of the space of the paths of the Markov process V by

$$F_f(V) = \int_0^\infty e^{-t} f(V(t))dt \tag{4}$$

Then, we study the weak convergence, as $n \to \infty$, of the finite-dimensional distributions of the sum

$$\Phi_n(f) = \sum_{i=1}^{n} \sigma_i Y_{n,i} F_f(V_i) \tag{5}$$

Clearly, if $\langle |f|, |f|\rangle$ is finite for all functions f from the Schwartz space of C^∞ functions that decrease at infinity faster than any polynomial, we can take the Schwartz space as the index set of Φ_n. This is possible, for example, when the Markov process V is Brownian motion.

We now define the limiting field. A random variable X is said to have a symmetric α-stable (SαS) distribution if its characteristic function is given by

$$\mathbf{E}\exp\{itX\} = \exp\{-\sigma^\alpha |t|^\alpha\} \tag{6}$$

The SαS distribution is thus characterized by index $\alpha, 0 < \alpha \leq 2$, and the scaling parameter σ. A stochastic process is SαS if all its finite-dimensional distributions are SαS, i.e., iff any linear combination of its components is an SαS random variable. On a separable finite measure space $(T, \mathcal{T}, \mu)$ let $X(\cdot)$ be a SαS Lévy random measure with control measure μ and Lévy measure $\nu(dx) = c_\alpha^{-1} dx/x^{-1-\alpha}$, where c^α is as in (2). This means that
(i) for every $A \in \mathcal{T}, X(A)$ is a SαS random variable with characteristic function

$$\begin{aligned}\mathbf{E}\exp\{itX(A)\} &= \exp\{-\mu(A)|t|^\alpha\}, \\ &\equiv \exp\left\{-\int_A \int_{R\setminus\{0\}} (1-\cos tx)\nu(dx)\mu(ds)\right\}\end{aligned} \tag{7}$$

(ii) $X(\cdot)$ is independently scattered and σ-additive, i.e., for disjoint sets $A_1, \ldots, A_k, \ldots$ in $\mathcal{T}$, the random variables $X(A_1), \ldots, X(A_k)$ are independent and $X(\cup_{j=1}^\infty A_j) = \sum_{j=1}^\infty X(A_j)$.

We will denote by Xf the integral of f with respect to measure X and by $X^k f$ the multiple integral of order k with respect to measure X. The construction of these is given in Kallenberg and Szulga (1989).

Now we choose a specific separable finite measure space $(T, \mathcal{T}, \mu)$. Let $T = D([0,\infty))$ denote the path space of the R^d-valued Markov process V, let $\mathcal{B}(T) = \mathcal{T}$ denote the Borel σ-algebra of the cylinder sets in T, and let μ denote the measure induced by the process V. Thus, $V, V_1, V_2, \ldots$ are i.i.d. random variables uniformly distributed on $(T, \mathcal{T}, \mu)$. Take $X(\cdot)$ to be a SαS Lévy random measure on $(T, \mathcal{T}, \mu)$ with control measure μ.

A family of random variables $\{\Phi(f),\ f \in S_d\}$ is called a generalized random field if the following two conditions hold:
(a) Φ is a linear random functional, i.e., $\Phi(af+bh) = a\Phi(f) + b\Phi(h)$ a.s. for all f, $h \in S_d$, and all $a, b \in R$;
(b) Φ has a version with values in the dual space S_d'.
Recall from (4) the definition of the functional F_f on space T. We now consider a family of random variables

$$\{\Phi(f),\ f \in S_d\} \overset{D}{=} \{\int_T F_f(u) X(du),\ f \in S_d\} \tag{8}$$

Proposition 1. *Let $0 < \alpha < 2$. Then (8) defines the generalized stable random field on S_d such that for every $f \in S_d$ and $t \in R$,*

$$\begin{aligned} \mathbf{E}\exp\{it\Phi(f)\} &= \exp\left\{-|t|^\alpha \int_T |F_f(u)|^\alpha \ \mu(du)\right\}, \\ &\equiv \exp\left\{-|t|^\alpha \mathbf{E}\left|F_f(V)\right|^\alpha\right\} \end{aligned} \tag{9}$$

Theorem 2. *Let $0 < \alpha < 2$. As $n \to \infty$, the finite-dimensional distributions of $\{\Phi_n(f),\ f \in S_d\}$ converge weakly to those of the stable random field $\{\Phi(f),\ f \in S_d\}$.*

In order to construct a kth order integral with respect to Lévy random measure X, we have to consider systems of k Markov processes. For a function f_k of k variables, define the functional

$$F_{f_k}(V_1, \ldots, V_k) := \int_0^\infty \cdots \int_0^\infty e^{-t_1-\ldots-t_k} f_k(V_1(t_1), \ldots, V_k(t_k)) dt_1 \ldots dt_k \tag{10}$$

Since

$$\mathbf{E}\left|F_{f_k}\left(V_{i_1}, \ldots, V_{i_k}\right)\right|^2 = \langle f_k, f_k \rangle, k \geq 1, \tag{11}$$

where

$$\langle f_k, h_k \rangle := \int_{R^{3dk}} m(d\mathbf{a}) f_k(\mathbf{x}) g^2(\mathbf{a}, \mathbf{x}) g(\mathbf{x}, \mathbf{y}) h_k(\mathbf{y}) d\mathbf{x}\, d\mathbf{y} \tag{12}$$

$$g(\mathbf{x}, \mathbf{y}) = g(x_1, y_1) \ldots g(x_k, y_k),\ g^2(\mathbf{a}, \mathbf{x}) = g^2(a_1, x_1) \ldots g^2(a_k, x_k) \tag{13}$$

(see Feldman and Rachev, 1993), we restrict our parameter set to functions from the space

$$S_d^k \equiv S_d^k(g) := \left\{f_k : f_k = f_k(x_1, \ldots, x_k) \text{ symmetric in } C^\infty(R^{dk}) \text{ with } \langle |f_k|, |f_k| \rangle < \infty\right\} \tag{14}$$

We are interested in studying the limiting distribution, as $n \to \infty$, of the normalized sum

$$\Psi_n(f_k) := \sum_{i \leq i_1 < \ldots < i_k \leq n} \sigma_{i_1} \ldots \sigma_{i_k} Y_{ni_1} \ldots Y_{ni_k} F_{f_k}(V_{i_1}, \ldots, V_{i_k}) \tag{15}$$

Proposition 3. *Let $0 < \alpha < 2$. The field*

$$\{\Psi(f_k), f_k \in S_d^k\} \stackrel{D}{=} \{X^k F_{f_k}, f_k \in S_k^d\}$$
$$\equiv \left\{ \int_T \cdots \int_T F_{f_k}(u_1, \ldots, u_k) X(du_1) \ldots X(du_k),\ f_k \in S_k^d \right\} \quad (16)$$

is well defined and is a generalized random field on S_d^k.

Theorem 4. *Let $0 < \alpha < 2$. As $n \to \infty$, the finite-dimensional distributions of $\left\{\Psi_n(f_k),\ f_k \in S_d^k\right\}$ converge weakly to those of $\left\{\Psi(f_k),\ f_k \in S_d^k\right\}$.*

3. PROOFS

We will now present the proofs of the results stated in Section 2. We start with the following lemma.

Lemma 5. *For $0 < \alpha < 2$ and every $f \in S_d$,*

$$\int_T |F_f(u)|^\alpha \mu(du) \le (\langle f, f \rangle)^{\alpha/2} < \infty \qquad (17)$$

Proof. Use Lyapunov's inequality (p. 191 of Shiryayev, 1984) and (11) to get

$$\int_T |F_f(u)|^\alpha \mu(du) \equiv \mathbf{E}\,|F_f(V)|^\alpha \le \left(\mathbf{E}\,|F_f(V)|^2\right)^{\alpha/2} = (\langle f, f \rangle)^{\alpha/2} < \infty$$

The proof is complete.

Proof of Proposition 1. The functional $F_f(V) = \int_0^\infty e^{-t} f(V(t))dt \equiv F(f, V)$ is jointly measureable on $S_d \times T$. Moreover, for each fixed $f \in S_d$, $F_f(V) \in L^\alpha(T)$ (by Lemma 5). Thus, (cp. Rosinski, 1986) the integral in (8) is well defined. Therefore Φ has the integral representation given by (8) and (cf., (3.2.2) Samorodnitsky and Taqqu, 1990), its distribution is specified by (9). It remains to be shown that Φ is a generalized field. Clearly, Φ is linear. By Corollary 4.2 of Walsh (1986), continuity in probability assures that the field has values in the dual space S_d'.

By Theorem 2, p. 254 of Shiryayev (1984), and the linearity of Φ, it is enough to show that

$$\pi_j(t) \to 0 \text{ in } S_d \text{ as } j \to \infty \Rightarrow E|\Phi(\pi_j)|^p \to 0, \quad p > 0. \qquad (18)$$

By property 1.2.10, p. 16 of Samorodnitsky and Taqqu (1990) for $0 < p < \alpha$

$$\mathbf{E}|\Phi(\pi_k)|^p = (c(p,\alpha))^p \left\{ \int_T |F_{\pi_k}(u)|^\alpha \mu(du) \right\}^{p\backslash\alpha}$$

where $c(p, \alpha)$ is a constant that depends on α and p, and hence does not affect the convergence. Thus (18) follows immediately from Lemma 5 and the proof is complete.

Proof of Theorem 2. By Proposition 3.1 in Szulga (1992) with $k = 1$, it is clear that as $n \to \infty$, $\Phi_n(f)$ converges weakly to the stable integral XF_f which in turn is equal, in distribution, to $\{\Phi(f),\ f \in S_d\}$ (see Proposition 1). Use the Cramér-Wold device (p. 49 of Billingsley, 1968) and the linearity of Φ_n and Φ to complete the proof.

For $f_k \in S_d^k$, define the functional $\tilde{F}_{f_k}(V_1, \ldots, V_k)$ as:

$$\tilde{F}_{f_k}(V_1, \ldots, V_k) = \begin{cases} F_{f_k}(V_1, \ldots, V_k) & \text{if all } k \text{ arguments are distinct} \\ 0 & \text{otherwise} \end{cases} \qquad (19)$$

Since the probability that any two or more of the paths of the Markov processes V_1, $V_2, \ldots, V_n$ are identical is zero, it follows that

$$\left\{\tilde{F}_{f_k}(V_1,\ldots,V_k),\, f_k \in S_d^k\right\} \stackrel{D}{=} \left\{F_{f_k}(V_1,\ldots,V_k),\, f_k \in S_d^k\right\} \tag{20}$$

Consider the set $\Omega_0 = \{\omega : V_1(\omega) \neq \ldots \neq V_k(\omega)\}$. Clearly, $P(\Omega_0) = 1$. Moreover, when restricted to the set Ω_0, the functionals $F_{f_k}(V_1,\ldots,V_k)$ and $\tilde{F}_{f_k}(V_1,\ldots,V_k)$ are identical. Therefore, for the next two results in this section, we will restrict our attention to Ω_0, and denote the functionals as $F_{f_k}(V_1,\ldots,V_k)$.

Proof of Proposition 3. The functional $F_{f_k}(V_1,\ldots,V_k)$ is jointly measurable on $S_d^k \times T^k$. Since the functions $f_k(u_1,\ldots,u_k) = f(u_1)\ldots f(u_k)$ are dense in S_d^k, it suffices to prove the Proposition for such f_k. Note that in this case

$$F_{f_k}(u_1,\ldots,u_k) = F_f(u_1)\ldots F_f(u_k) \tag{21}$$

Following Kallenberg and Szulga (1989), we set ζ to be a Poisson process on $R\backslash\{0\}\times T$, with intensity $\nu \times \mu$, so that ζ is constructed from the jumps of the process X. We denote its symmetrized version by $\tilde{\zeta}$. Then the following representation of the integral holds

$$X^k F_{f_k} = \tilde{\zeta}^k(LF_{f_k}) \text{ a.s.} \tag{22}$$

Here L is the operator defined on the space of functionals on T^k as

$$LF_{f_k} \equiv LF_{f_k}(u_1,\ldots,u_k;x_1,\ldots,x_k) = x_1\ldots x_k F_{f_k}(u_1,\ldots,u_k),$$

where $x_i \in R\setminus\{0\}$, $u_i \in T$, $i = 1,\ldots,k$ and the integral on the right of (22) exists if F_{f_k} belongs to the class $\mathcal{L}$, where

$$\mathcal{L} = \left\{F_{f_k} : \zeta^k(LF_{f_k})^2 < \infty \ \text{ a.s.}\right\}.$$

Let $\zeta_1,\ldots,\zeta_k$ be independent copies of ζ and note that by Theorem 3.4 in Kallenberg and Szulga (1989), the condition $\zeta^k(LF_{f_k})^2 < \infty$ a.s. is equivalent to $\zeta_1\ldots\zeta_k(LF_{f_k})^2 < \infty$ a.s. By the independence of $\zeta_1,\ldots,\zeta_k$, (21), and Lemma 2.2 in Kallenberg and Szulga (1989), this is equivalent to

$$\int_T \int_{R\backslash\{0\}} \left((x^2 F_f^2)\wedge 1\right) \nu(dx)\mu(du) < \infty \ \text{ a.s.}$$

which in its turn follows if the following two conditions hold:

$$\text{(i)} \int_T \int_{R\backslash\{0\}} \left((x^2 \wedge 1) F_f^2\right) \nu(dx)\mu(du) < \infty \ \text{ a.s.}$$

$$\text{(ii)} \int_T \int_{R\backslash\{0\}} \left((x^2 \wedge 1)(F_f^2 \vee 1)\right) \nu(dx)\mu(du) < \infty \ \text{ a.s.}$$

Since $\int_{R\backslash\{0\}}(x^2 \wedge 1)\nu(dx) < \infty$ for Lévy measure ν on $R\setminus\{0\}$ and (11) holds, (i)-(ii) are satisfied. Thus, the multiple integral $X^k F_{f_k}$ exists. Linearity of Ψ in f_k follows from the linearity of F_{f_k}. To show that Ψ has a version in $(S_d^k)'$ follow the proof of Proposition 1, keeping in mind that (cf. (1.6) of Kallenberg and Szulga, 1989):

$$E(\tilde{\zeta}^k(LF_{f_k}))^2 = \sum_{m=1}^{k} k! \binom{k}{m}^2 \left\{\nu^m\left(\nu^{k-m}(x_1\ldots x_k)\right)^2\right\}\left\{\mu^m\left(\mu^{k-m}\left(F_f(u_1)\ldots F_f(u_k)\right)\right)^2\right\}$$

where $\mu(F_f(u))^2 \equiv \int_T |F_f(u)|^2 \mu(du) = \langle f, f\rangle < \infty$. The Proposition is proved.
Proof of Theorem 4. It is clear from Proposition 3.1 in Szulga (1992), the Cramér-Wold device, and the linearity of $\Psi_n(f_k)$ that the finite-dimensional distributions of the field $\{\Psi_n(f_k),\ f_k \in S_d^k\}$, converge weakly to those of $\{X^k \tilde{F}_{f_k},\ f_k \in S_d^k\}$, as $n \to \infty$. The proof now follows immediately from (20) and Proposition 3.
Remark. The above results can be extended in two directions. The first one, which is quite trivial, is to introduce the parameter $\theta > 0$, to substitute the Green's function g^θ for g wherever it appeared, and to change the exponent e^{-t} to $e^{-\theta t}$ in the definitions (4) and (10). We will denote the resulting functionals by F_f^θ and the spaces of the test functions by $S_d^{\theta,k}$, $S_d^{\theta,1} \equiv S_d^\theta$. It is clear that all previous results hold under these changes; we will call the limiting families of fields $\{\Phi^\theta, \theta > 0\}$ and $\{\Psi^\theta, \theta > 0\}$.

The above results also hold for the case where the fields are indexed by measures. Let us briefly explain how to construct $F_{\gamma_k}^\theta \equiv F_{\gamma_k}^\theta(X_{i_1}, \dots, X_{i_k})$ when γ_k is a symmetric measure on R^{dk}, $k \geq 1$, such that

$$\langle \gamma_k, \gamma_k \rangle_\theta = \int m(d\mathbf{a}) g^{2\theta}(\mathbf{a}, \mathbf{x}) g^\theta(\mathbf{x}, \mathbf{y}) \gamma_k(d\mathbf{x}) \gamma_k(d\mathbf{y}) < \infty. \tag{23}$$

We denote the class of such measures by $\mathcal{M}^{\theta,k} = \mathcal{M}^k(g^\theta)$, $\mathcal{M}^{\theta,1} \equiv \mathcal{M}^\theta$. Theorem 2.1 in Adler and Epstein (1987) guarantees existence of the functional $F_{\gamma_k}^\theta$ for each $\gamma_k \in \mathcal{M}^{\theta,k}$. If γ_k is absolutely continuous with respect to Lebesgue measure with density f_k then $F_{\gamma_k}^\theta = F_{f_k}^\theta$. Otherwise, it is constructed as the L^2 limit of path integrals of the form $F_{\gamma_k,\delta}^\theta := \int_{R_+^k} e^{-\theta(t_1+\dots+t_k)} b_{\gamma_k,\delta}^\theta(V_1(t_1), \dots, V_k(t_k)) dt_1 \dots dt_k$. Here, $\gamma_\delta^\theta(d\mathbf{x}) = b_{\gamma_k,\delta}^\theta(\mathbf{x}) d\mathbf{x}$, and $b_{\gamma_k,\delta}^\theta(\mathbf{x}) = \int_{R^{kd}} e^{-\theta(\delta_1+\dots+\delta_k)} p_{\delta_1}(x_1, y_1) \dots p_{\delta_k}(x_k, y_k) \gamma_k(d\mathbf{y})$. Thus, γ_δ^θ is a smoothed version of γ_k.

4. POINT-INDEXED STABLE RANDOM FIELDS

In the previous section we have shown the construction of the stable random fields indexed by functions $f \in S_d$. We will now show how the same construction allows us to obtain stable fields on R^d. To do so, we would like to apply Theorem 2 to the case of measure $\gamma_x(dz) = \delta_x(z)dz$, where $x \in R^d$. Of course, this is possible only when $\gamma_x \in \mathcal{M}^1$, i.e., when (23) holds. This is true, for example, for Brownian motion on R^1. The corresponding functional $F_{\gamma_x}(V)$ is then the exponentially weighted local time $L_x(V)$ of the process V at point x.

Denote by Φ_x, $x \in R^d$, the stable random field obtained by applying the measure-variant of Theorem 2 to the sums of the functional $L_x(V)$. Then Φ_x, $x \in R^d$, has the integral representation

$$\Phi(x) \equiv \Phi_x = \int_T L_x(u) X(du).$$

So, whenever the local time of the Markov process exists, we obtain the point-indexed stable random field $\{\Phi(x),\ x \in R^d\}$, $x \in R^d$, with finite-dimensional distributions given by

$$\mathbf{E} \exp\{i \sum_{i=1}^n t_i \Phi_{x_i}\} = \exp\left\{-\mathbf{E}\left|\sum_{i=1}^n t_i L_{x_i}(V)\right|^\alpha\right\}$$

Clearly, when a point-indexed field $\{\Phi(x),\ x \in R^d\}$ exists, we can always create a S_d indexed version of it by setting $\Phi(f) = \int_{R^d} \Phi(x) f(x) dx$.

In a similar manner, for those processes for which measure $\gamma_2(dx_1, dx_2) = \delta_x(x_1)\delta(x_1 - x_2)dx_1 dx_2$ is of finite norm, one can use exponentially weighted intersection local times $L_x(V_1, V_2)$ to define the point-indexed random fields $\{\Psi(x),\ x \in R^d\}$ as limits of sums $\sum_{1\le i_1<i_2\le n} \sigma_{i_1}\sigma_{i_2} Y_{n,i_1} Y_{n,i_2} L_x(V_{i_1}, V_{i_2})$. By Theorem 4

$$\Psi(x) \overset{D}{=} \int_T \int_T L_x(u_1, u_2) X(du_1) X(du_2).$$

5. RENORMALIZABLE STABLE FIELDS

The concept of renormalizability of families of generalized random fields was introduced in Adler and Epstein (1987) as an extension of the property of self-similarity. There the authors considered conditions for existence of this property for some Gaussian families and their multiple integrals. Here we modify their definition of renormalizability to include rescaling of an additional parameter and study this property for the families of stable fields that are obtained as limits of functionals of the paths of self-similar Markov processes.

In the following, we will assume that the initial measure of the Markov process is

$$m(dx) = \frac{1}{|B|} 1_B(x) dx \tag{24}$$

where $B \subset R^d$ is a fixed set of finite Lebesgue measure $|B|$.

Definition. *A process V is said to be self-similar with index β, if for any $\eta > 0$, V and the process*

$${}^{\beta}_{\eta}V(t) := \eta^{-1}\tilde{V}(\eta^{\beta} t) \tag{25}$$

are identical in distribution, where $\tilde{V}$ is a process (on another probability space) which has the same transition probabilities as V but initial measure $\tilde{m}(dx) = \frac{1}{|\eta B|} 1_{\eta B}(x) dx$.

Note that the initial measure has to be changed and the set B has to be scaled in order to preserve the distribution of the starting points after scaling $\tilde{V}$ by the factor η. For any function $f(\mathbf{x})$ on R^{dk} and $\tau > 0$, define

$${}^{\tau}_{\eta}f(\mathbf{x}) := \eta^{-\tau} f(\eta^{-1}\mathbf{x}). \tag{26}$$

Lemma 6. *Let V be a Markov process which is self-similar with index β, and let $q \in S_d^{\theta,k}$, $k \ge 1$. Then*

$$F_q^{\theta}(V_1, \dots, V_k) \overset{D}{=} F^{\theta\eta^{-\beta}}_{{}^{\tau}_{\eta}q}\left(\tilde{V}_1, \dots, \tilde{V}_k\right), \quad \tau = k\beta \tag{27}$$

Proof. Using the self-similarity property of V, and then making the change of variables $\eta^{\beta} t_i = s_i$, we get

$$\begin{aligned} F_q^{\theta}(V_1, \dots, V_k) &= \int_{R_+^k} e^{-\theta(t_1+\dots+t_k)} q(V_1(t_1), \dots, V_k(t_k)) dt_1 \dots dt_k \\ &\overset{D}{=} \int_{R_+^k} e^{-\theta(t_1+\dots+t_k)} q(\eta^{-1}\tilde{V}_1(\eta^{\beta}t_1), \dots, \eta^{-1}\tilde{V}_k(\eta^{\beta}t_k)) dt_1 \dots dt_k \\ &= \int_{R_+^k} e^{-\theta\eta^{-\beta}(s_1+\dots+s_k)} q(\eta^{-1}\tilde{V}_1(s_1), \dots, \eta^{-1}\tilde{V}_k(s_k)) \eta^{-k\beta} ds_1 \dots ds_k \\ &\overset{D}{=} F^{\theta\eta^{-\beta}}_{{}^{\tau}_{\eta}q}\left(\tilde{V}_1, \dots, \tilde{V}_k\right), \end{aligned} \tag{28}$$

with $\tau = k\beta$. This proves the lemma.

Let us denote by $X^m(.)$, the SαS Lévy random measure $X(.)$, on $(T, \mathcal{T}, \mu^m)$, with control measure μ^m. The index m indicates that the process V starts according to the initial measure m. Further, let $X^{m,k}f$ denote the multiple integral of order k, of the function f with respect to $X^m(.)$. When $k = 1$, we set $X^{m,1}f \equiv X^m f$. We define the following two families of stable fields:

$$\Phi^{\theta,m}(q) \stackrel{D}{=} \int_T F_q^\theta(u) X^m(du) \tag{29}$$

$$\Phi^{\theta\eta^{-\beta}}\left(\begin{matrix}\tau\\ \eta\end{matrix} q\right) \stackrel{D}{=} \int_T F^{\theta\eta^{-\beta}}_{{}^{\tau}_{\eta}q}(u) X^{\tilde{m}}(du) \tag{30}$$

Definition. *The family of random fields $\{\Phi^{\theta,m}, \theta > 0\}$ is renormalizable with renormalizing parameters (τ, ρ) if, for every $\eta > 0$,*

$$\Phi^{\theta,m} \stackrel{D}{=} {}^{\tau}_{\eta}\Phi^{\theta\eta^{-\rho},\tilde{m}} \tag{31}$$

where

$${}^{\tau}_{\eta}\Phi^{\theta\eta^{-\rho},\tilde{m}}(q) = \Phi^{\theta\eta^{-\rho},\tilde{m}}\left(\begin{matrix}\tau\\ \eta\end{matrix} q\right), \quad {}^{\tau}_{\eta}q(x) := \eta^{-\tau} q(\eta^{-1}x) \tag{32}$$

If the $\Phi^{\theta,m}$ are measure indexed, set

$${}^{\tau}_{\eta}\Phi^{\theta\eta^{-\rho},m}(\gamma) = \Phi^{\theta\eta^{-\rho},m}\left(\begin{matrix}\tau\\ \eta\end{matrix}\gamma\right), \quad {}^{\tau}_{\eta}\gamma(A) := \eta^{-\tau+dk}\gamma(\eta^{-1}A), \quad \gamma \in \mathcal{M}^{\theta,k}. \tag{33}$$

Theorem 7. *Let $(V(t), m)$ be a self-similar Markov process with index β, and let g^θ, $\theta > 0$, be the corresponding Green function. The family of stable random fields $\{\Phi^{\theta,m}, \theta > 0\}$ is renormalizable with parameters (β, β).*

Proof. Using (9) and Lemma 6 with $\tau = \beta$, one obtains:

$$\begin{aligned}
\mathbf{E}\exp\left\{it\Phi^{\theta\eta^{-\beta},\tilde{m}}\left(\begin{matrix}\beta\\ \eta\end{matrix} q\right)\right\} &= \exp\left\{-|t|^\alpha \mathbf{E}\left|F^{\theta\eta^{-\beta}}_{{}^{\beta}_{\eta}q}(\tilde{V})\right|^\alpha\right\}\\
&= \exp\left\{-|t|^\alpha \mathbf{E}\left|F^{\theta}_{q}(V)\right|^\alpha\right\}\\
&= \mathbf{E}\exp\left\{it\Phi^{\theta,m}(q)\right\}, \quad t \in R
\end{aligned}$$

and the proof is complete.

Our next result establishes the renormalizability of the family $\{\Psi^{\theta,m}(f_k), \theta > 0\}$ of multiple integrals of the fields $\{\Phi^{\theta,m}, \theta > 0\}$.

Theorem 8. *Let $k > 1$. The family $\{\Psi^{\theta,m}, \theta > 0\}$ is renormalizable with parameters $(k\beta, \beta)$.*

Proof. Let "$\Rightarrow$" denote weak convergence. By Theorem 4,

$$\sum_{1 \le i_1 < \ldots < i_k \le n} \sigma_{i_1} \ldots \sigma_{i_k} Y_{ni_1} \ldots Y_{ni_k} F^\theta_{q_k}(V_{i_1}, \ldots, V_{i_k}) \Rightarrow \Psi^{\theta,m}(q_k) \;\; \text{as } n \to \infty,$$

$$\sum_{1 \le i_1 < \ldots < i_k \le n} \sigma_{i_1} \ldots \sigma_{i_k} Y_{ni_1} \ldots Y_{ni_k} F^{\theta\eta^{-\beta}}_{{}^{k\beta}_{\eta}q_k}(\tilde{V}_{i_1}, \ldots, \tilde{V}_{i_k}) \Rightarrow \Psi^{\theta\eta^{-\beta},\tilde{m}}\left(\begin{matrix}k\beta\\ \eta\end{matrix} q_k\right) \;\; \text{as } n \to \infty.$$

By Proposition 3, $\Psi^{\theta,m}(q_k) \stackrel{D}{=} X^{m,k}F^\theta_{q_k}$, $\Psi^{\theta\eta^{-\beta},\tilde{m}}\left(\begin{matrix}k\beta\\ \eta\end{matrix} q_k\right) \stackrel{D}{=} X^{\tilde{m},k}F^{\theta\eta^{-\beta}}_{{}^{k\beta}_{\eta}q_k}$ and by Lemma 6, $F^\theta_q(V_1, \ldots, V_k) \stackrel{D}{=} F^{\theta\eta^{-\beta}}_{{}^{k\beta}_{\eta}q}(\tilde{V}_1, \ldots, \tilde{V}_k)$ Thus, we have.

$$\Psi^{\theta,m}_{q_k} \stackrel{D}{=} \Psi^{\theta\eta^{-\beta},\tilde{m}}_{{}^{k\beta}_{\eta}q_k}$$

and the theorem is proved.

6. SIMULATION OF STABLE RANDOM FIELDS

Here we provide graphs of some R^d-indexed stable fields for $d = 1$ and 2. The simulations were based on results of Theorem 2 and discussion in Section 4, which allow one to build the fields Φ_x from the sums of functionals of the Markov processes V. We would like to explain briefly how these simulations were done.

We simulate symmetric random walks S_k on the integer lattice Z^d, $d = 1, 2$, i.e.,

$$S_0 = 0, \quad S_k = \xi_1 + \ldots + \xi_k, \; k = 1, 2, \ldots,$$

where, the ξ_i are i.i.d. Rademacher random variables. Their initial measure is $m(dx) = \mathbf{1}_B(x)dx$, where $B \subset R^d$ is a fixed set of finite Lebesgue measure $|B|$ to be specified later. Following a discussion in Dynkin (1988), use the functional of the random walk

$$F_{l,m}(S) = \frac{1}{m}\sum_{i=1}^{mT} e^{-i/m}\delta_l\left(\frac{1}{\sqrt{m}}S_i\right), \; l \in \frac{1}{\sqrt{m}}Z^d \tag{34}$$

to approximate the value of the exponentially weighted local time L_l of the Brownian motion (which exists for $d = 1$). This is a good choice for several reasons: As $m \to \infty$, the finite-dimensional distributions of the scaled random walk $\{S_{mt}/\sqrt{m}, t \in \{0, 1/m, 2/m, \ldots\}\}$ on the lattice $Z^d/\sqrt{m}$ converge to those of the Brownian motion on R^d. We would like to have a functional of $S/\sqrt{m}$ which approximates the functional of Brownian motion

$$F_f(V) = \int_0^\infty e^{-t} f(V(t))dt. \tag{35}$$

Since the value e^{-20} is less than 10^{-9}, for computational purposes we restrict the interval of integration in (35) to $[0, T]$, with $T = 20$. Dynkin (1988) gives the approximation as

$$F_{f,m}(S) = \frac{1}{m}\sum_{i=1}^{Tm} e^{-i/m} f(S_i/\sqrt{m}).$$

In order to obtain point-indexed fields, we replace the function f by the Dirac delta-function at l. Finally, we use sums

$$\Phi_n(l) := \sum_{i=1}^{n} \sigma_i Y_{n,i} F_{l,m}(S),$$

to approximate values of the stable field $\Phi(l)$.

For the case $d = 1$, we have chosen $m = 1000$, $n = 2000$, and $B = [-1000, 1000]$. We circularized the interval, i.e., when the random walk wanders off the left hand-side of $[-1000, 1000]$, it immediately reappears at the right hand-side and vice-versa. To fix the value of m we first obtained graphs of (34) for one fixed realization of the random walk (but different values of m) and concluded that the graph does not change much for $m > 1000$. To choose the value of n we fixed $\alpha = 1.95$ and $m = 1000$ and generated independently $n = 1500$ values of the stable process at a particular point. This histogram was very similar to the histograms of 1500 values from a SαS distribution with $\alpha = 1.95$ and $\sigma = 0.014678$ generated by the software package Splus. We also estimated the values of the parameters α and σ using a method based on McCullough (1986). The value of α was close to 1.95 and $\sigma = 0.014678$.

For $d = 2$ we took $m = 1500$, $n = 4000$ and $B = [0, 100) \times [0, 100)$. As for $d = 1$, we circularized the region.

To generate the stable random variates $Y_{n,i}$ that satisfy (2) we followed the algorithm in Bratley, Fox and Schrage (1987), which generates SαS random variates with scaling factor $\sigma = 1$, and then multiplied the values by $n^{-1/\alpha}$. The value of α is chosen before simulating the stable and Rademacher random variates.

The results of our simulations are presented in Figures 6.1-6.6. Figures 6.1-6.3 present SαS processes $\Phi(x)$, $x = l/m$, $l = -1000, -999, \ldots, 1000$ for three values of the parameter $\alpha = 1.1, 1.5$ and 1.95. The points $l = 0$ and 2001 on the graphs in the Figures correspond to the points $l = -1000$ and $x = 1000$, respectively. The graphs of the stable fields $\Phi(x)$, $x \in [0,100) \times [0,100)$ for $\alpha = 1.1$, 1.5 and 1.95 are given in Figures 6.4, 6.5, and 6.6. Each of these figures contains three graphs. The graph at the top is the graph of the stable random field. The graph at the bottom left is of the positive values of the field plotted separately; and the graph at the bottom right is of the negative values of the field. It is clear from the graphs that when α is close to 1, the field is fairly flat; as α increases the fields become more peaked.

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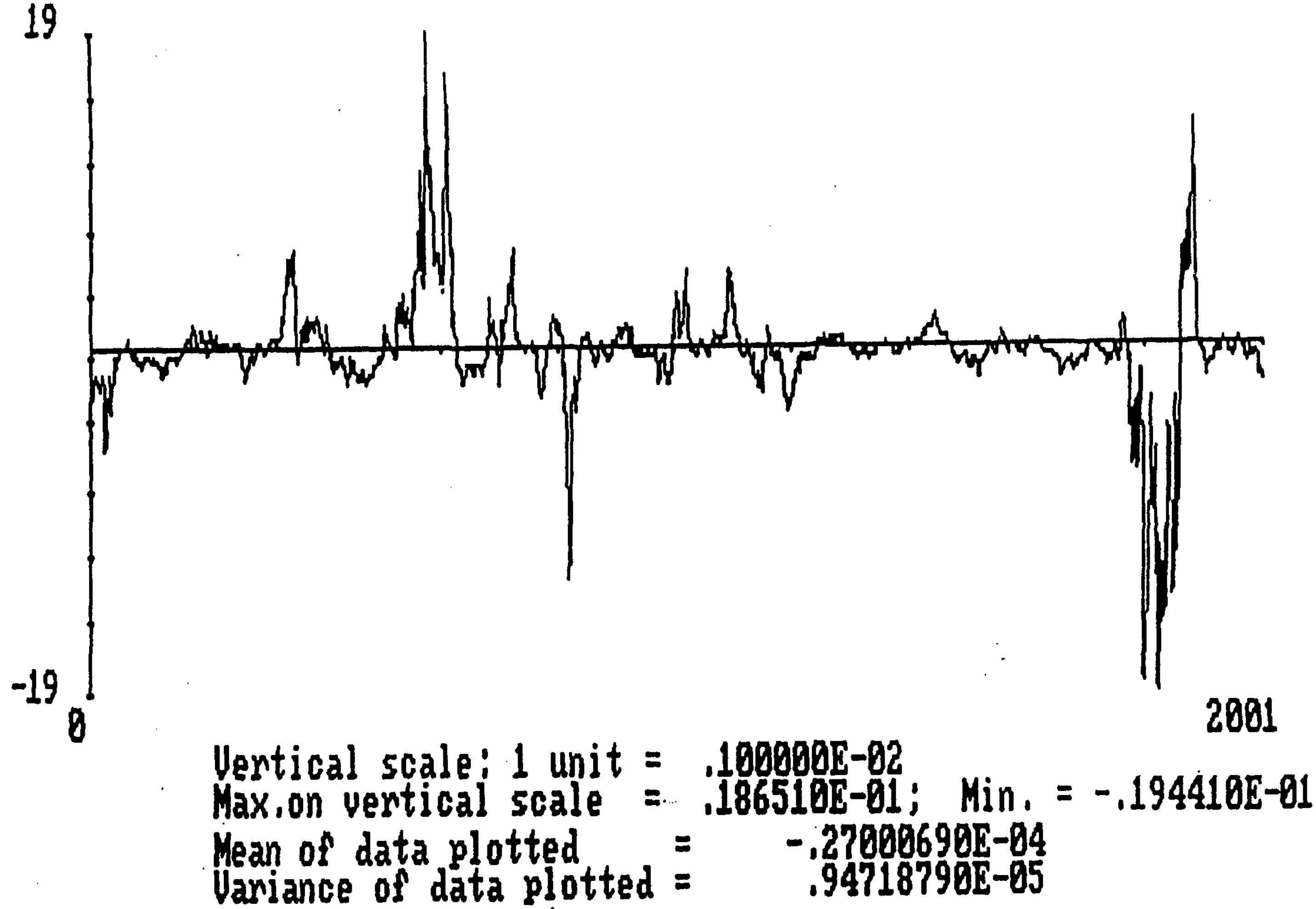
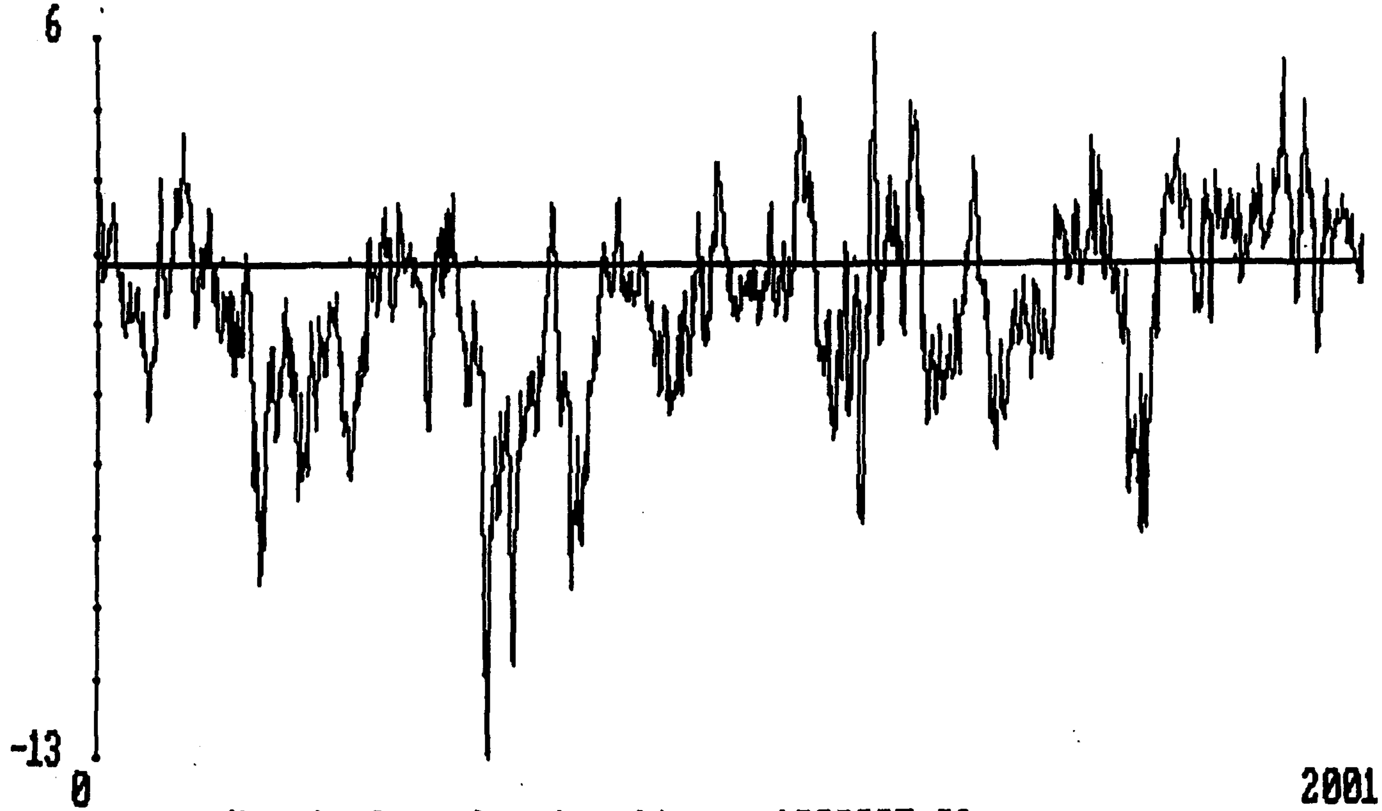
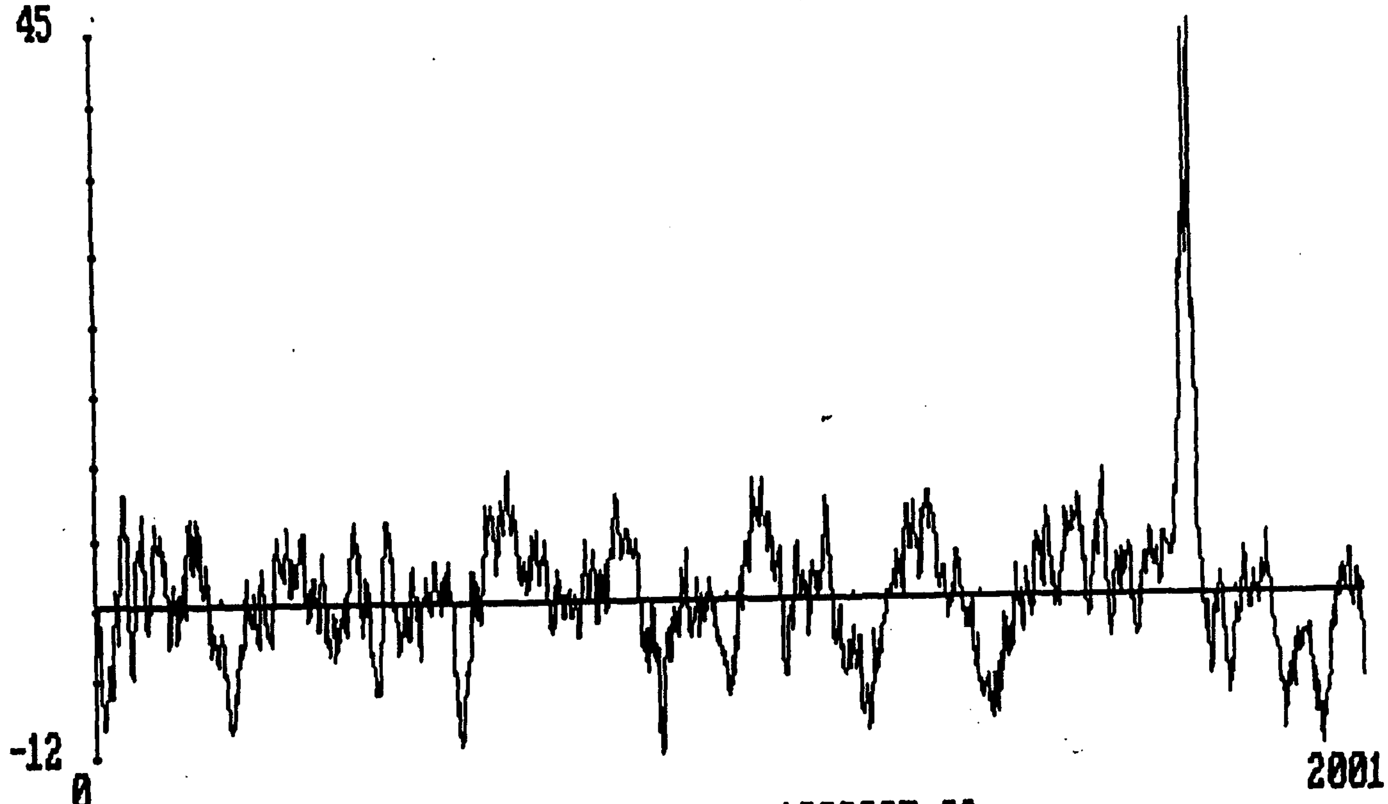


Fig. 6.1. Stable Process, $\alpha = 1.1$



Vertical scale: 1 unit = .100000E-02
Max.on vertical scale = .585700E-02; Min. = -.126110E-01
Mean of data plotted = -.10261070E-02
Variance of data plotted = .55368420E-05

Fig. 6.2. Stable Process, $\alpha = 1.5$



Vertical scale: 1 unit = .100000E-02
Max.on vertical scale = .451260E-01; Min. = -.118340E-01
Mean of data plotted = .54968100E-03
Variance of data plotted = .32296620E-04

Fig. 6.3. Stable Process, $\alpha = 1.95$

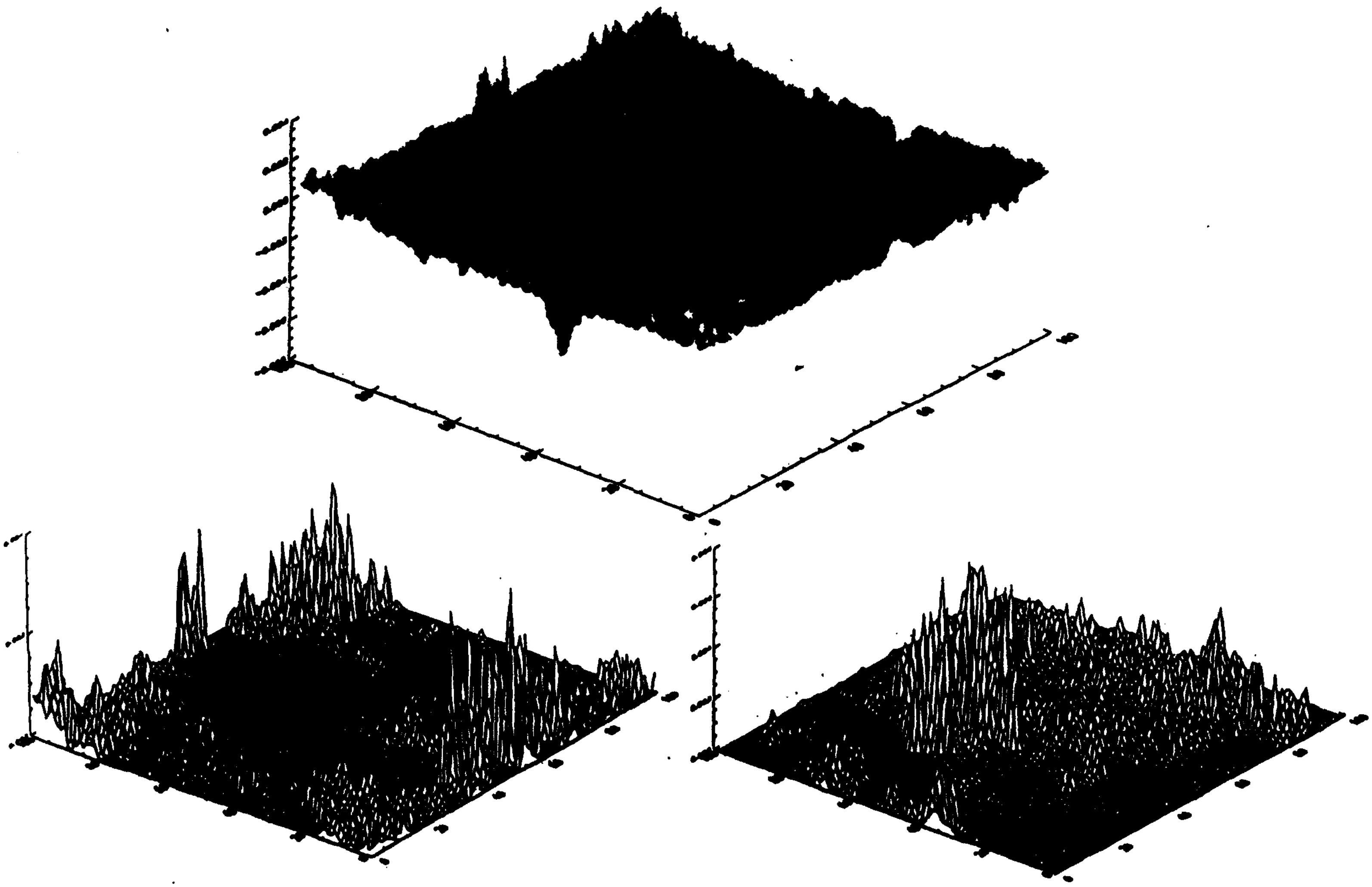


Fig. 6.4. Stable Random Field, $\alpha = 1.1$: (a) field, (b) positive values of the field, (c) negative values of the field

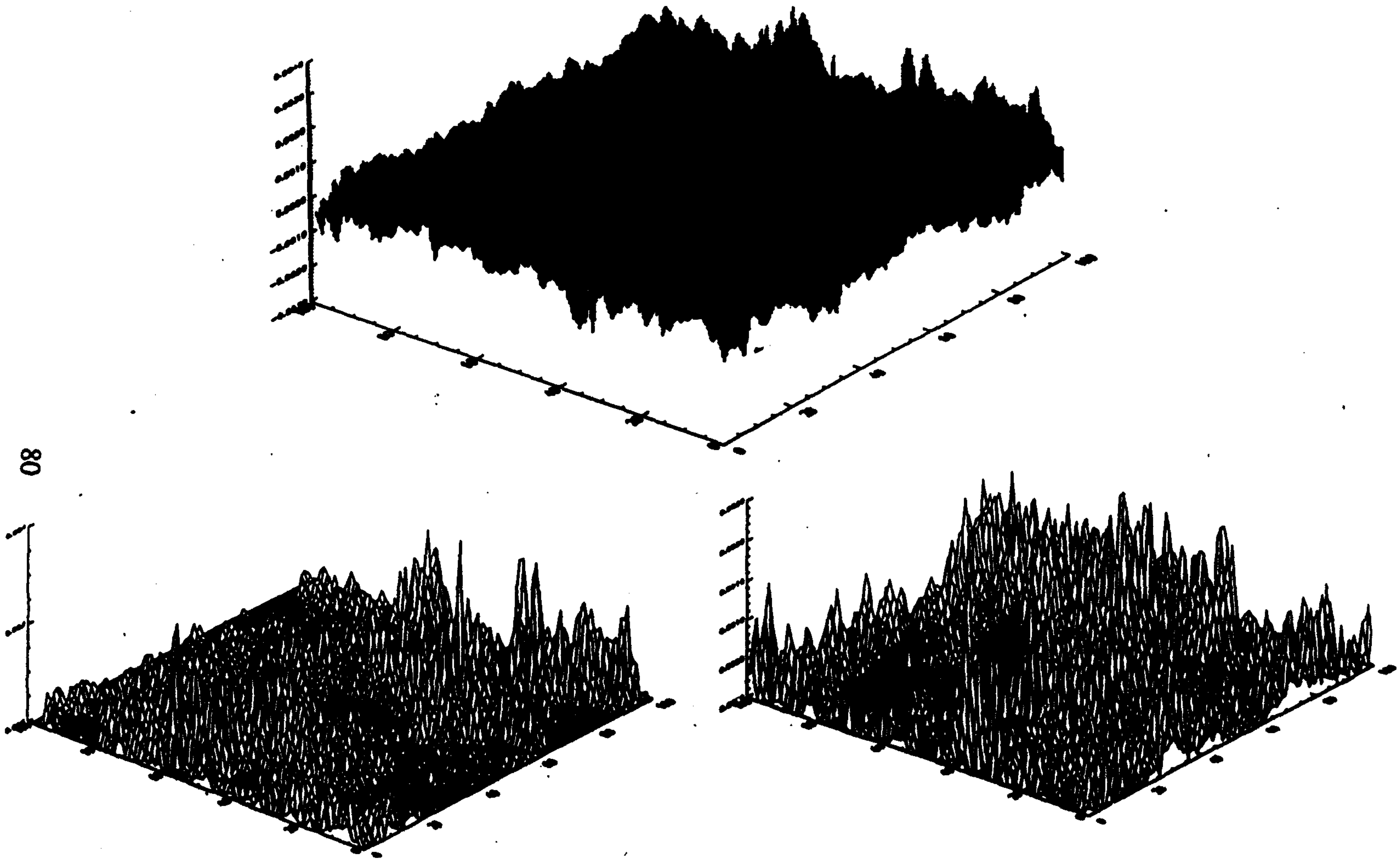


Fig. 6.5. Stable Random Field, $\alpha = 1.5$: (a) field, (b) positive values of the field, (c) negative values of the field

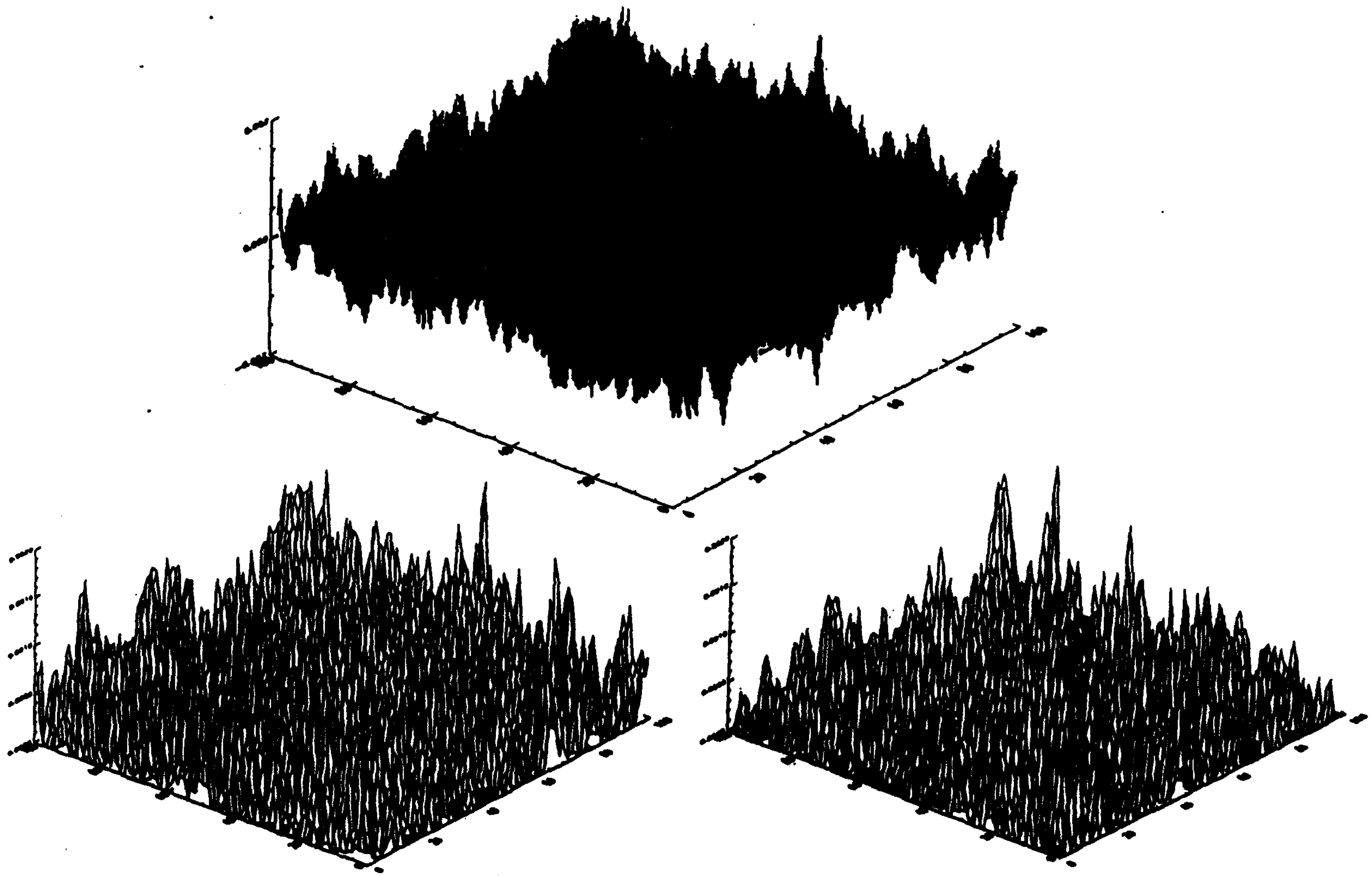


Fig. 6.8. Stable Random Field, $\alpha = 1.95$: (a) field, (b) positive values of the field, (c) negative values of the field

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